1. Some preliminaries...

1.1. Quantifiers. There are two symbols that will commonly appear in the textbook and throughout the course: $\forall$ and $\exists$. You can translate these symbols into English as follows:

$$\forall = \text{for every}$$

$$\exists = \text{there exists}$$

Example: Consider the following two sentences

The hotel has a gym, a pool, tennis courts, comfortable beds, and other athletic facilities
The hotel has a comfortable beds, as well as a gym, a pool, tennis courts, and other athletic facilities

The difference between these two statements should be clear, as well as the lesson to be learned: **The order of quantifiers matters!** Here is a math example:

A function $f : [a, b] \to \mathbb{R}$ *is uniformly continuous if and only if* $\forall\, \epsilon > 0\, \exists\, \delta > 0$ *such that* $\forall\, x, y \in [a, b]$ *with* $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$

Compared to:

A function $f : [a, b] \to \mathbb{R}$ *is continuous if and only if* $\forall\, \epsilon > 0$, $\forall\, x \in [a, b]\, \exists\, \delta > 0$ *such that* $\forall\, y \in [a, b]$ *with* $|x - y| < \delta \Rightarrow |f(x) - f(y)| < \epsilon$

1.2. Sets. A set is, generally speaking, is just a collection of objects. These objects could be points, numbers, or even collections of other sets! In this course you will mainly be concerned with sets of points in $\mathbb{R}^n$. To specify a set, it is common to use the following notation:

$$S = \{\text{Points from a particular space} \,|\, \text{that satisfy the following list of properties}\}$$

An important skill to have is to show that two sets S_1 and S_2 are equal; this is usually done by showing $S_1 \subseteq S_2$ and $S_2 \subseteq S_1$.
Group exercise: Find an example of an infinite collection of closed subsets $S_1, S_2, \dots \subseteq \mathbb{R}^n$ whose union is not closed.
Solution: Let $S_k = \overline{B(0, 1 - \frac{1}{k})} = \{x \in \mathbb{R}^n \,|\, |x| \le 1 - \frac{1}{k}\}$. I claim that:

$$\bigcup_{k=1}^{\infty} S_k = B(0, 1)$$

Which is not closed, since it does not contain its boundary. Pick any $x \in \bigcup_k S_k$, then by definition there exists $j \in \mathbb{N}$ such that $x \in \overline{B(0, 1 - \frac{1}{j})}$. This implies that $|x| \le 1 - \frac{1}{j} < 1$, so $x \in B(0, 1)$. This has shown that $\bigcup_k S_k \subseteq B(0, 1)$. Conversely, suppose that $x \in B(0, 1)$, then $|x| < 1$. This means that

$$d = \inf_{y \in \partial B(0,1)} |x - y| > 0$$

So, pick $N \in \mathbb{N}$ large enough that $\frac{1}{N} < d$. Now $x \in \overline{B(0, 1 - \frac{1}{N})}$, so $x \in \bigcup_k S_k$. This completes the proof.

1.3. Reading and writing math.

- **Reading mathematics is not the same as reading a novel**. When reading novels, one tends to skim and miss some details. These might be important or they might not be, but generally speaking it will not impact your understanding of the story. In mathematics, the complete opposite is true. A well written mathematics proof includes only exactly what needs to be in text in order to have a cohesive argument. Missing any one point likely means that you will not understand anything that follows. The key to understanding written mathematics is to take your time and understand each idea as it is presented to you. Force yourself to see the logical connection between previous steps and the current one, and try to guess what might come next before reading it.

- **The people grading your tests and assignments are not robots**. Sprinkle some English sentences into your proofs to explain the idea of what you're doing. It is difficult for graders to digest a wall of symbols without any explanation. The opposite is true as well; do not present a wall of text. Say only exactly what you need to say in order to convey the skeleton of your logical argument.

Class exercise: State the mean value theorem.
Solution: If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then $\exists\, c \in [a, b]$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

- **Mathematical statements take the form: If CONDITIONS then CONSEQUENCES**. Some common mistakes I have seen on tests:
  (1) Not giving any conditions. It is wrong to just simply write:

  $$f'(c) = \frac{f(b) - f(a)}{b - a}$$

  What is f? What are c, b, and a? The symbols are meaningless until you attach meaning to them using language.
  (2) Misstating the conditions, or omitting some conditions. In the example above, the statement doesn't make sense if f is not differentiable on (a, b) because $f'(c)$ does not exist. If continuity were omitted, then the right hand side could be undefined (e.g. $f(x) = \arctan(x)$ on $[-\pi/2, \pi/2]$ is not continuous.)
  (3) Stating the wrong consequences. Consider for example:
  If $f : [a, b] \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then $\forall\, c \in [a, b]$

  $$f'(c) = \frac{f(b) - f(a)}{b - a}$$

  This statement says that every differentiable function is the constant function zero. This is obviously crazy.